Defective Eigenvalues

Here is an alternative way to find generalized eigenvectors for a defective eigenvalue.

__Ex:__ $\underline{x}' = \begin{bmatrix} 0 & 1 & 2 \\ -5 & -3 & -7 \\ 1 & 0 & 0 \end{bmatrix} \underline{x}$

Characteristic eqn:

expand across bottom row $\rightarrow$ $\begin{vmatrix} -\lambda & 1 & 2 \\ -5 & -3-\lambda & -7 \\ 1 & 0 & -\lambda \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & 2 \\ -3-\lambda & -7 \end{vmatrix} + 0 + -\lambda \begin{vmatrix} -\lambda & 1 \\ -5 & -3-\lambda \end{vmatrix}$

$= -7 - 2(-3-\lambda) - \lambda\left(-\lambda(-3-\lambda) - 1(-5)\right)$

$= -7 + 6 + 2\lambda - \lambda(3\lambda + \lambda^2 + 5)$

$= -1 + 2\lambda - 3\lambda^2 - \lambda^3 - 5\lambda$

$= -\lambda^3 - 3\lambda^2 - 3\lambda - 1$

$= -(\lambda+1)^3 = 0$

$\lambda = -1$ with alg. mult $= 3$

Look for eigenvector for $\lambda = -1$

$\begin{bmatrix} 1 & 1 & 2 \\ -5 & -2 & -7 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow v_1 + v_3 = 0$

$\boxed{v_3 = -v_1}$

$v_1 + v_2 + 2v_3 = 0$

$v_1 + v_2 + 2(-v_1) = 0$

$v_2 - v_1 = 0$

$\boxed{v_2 = v_1}$

$-5v_1 - 2v_2 - 7v_3 = 0$

$-5v_1 - 2(v_1) - 7(-v_1) = 0$

$-7v_1 + 7v_1 = 0$

$0 = 0$

So v_1 is a free variable

so $\boxed{\text{geom. mult} = 1}$

$v = \begin{bmatrix} v_1 \\ \cdot \end{bmatrix} \quad \begin{bmatrix} 1 \\ \cdot \end{bmatrix}$

$$\underline{v} = \begin{bmatrix} v_1 \\ v_1 \\ -v_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

so $\boxed{\text{geom. mult} = 1}$

The defect of $\lambda = -1$ is $d = \text{alg mult} - \text{geom mult}$

$d = 3 - 1$

$\boxed{d = 2}$

So we need to find 2 generalized eigenvectors that satisfy:

start here

$(\underline{\underline{A}} + \underline{\underline{I}})^3 \underline{u}_3 = 0 \leftarrow (\underline{\underline{A}} + \underline{\underline{I}}) \underline{u}_3 = \underline{u}_2$

$(\underline{\underline{A}} + \underline{\underline{I}})^2 \underline{u}_2 = 0 \leftarrow (\underline{\underline{A}} + \underline{\underline{I}}) \underline{u}_2 = \underline{u}_1$

$(\underline{\underline{A}} + \underline{\underline{I}}) \underline{u}_1 = 0 \leftarrow$ instead of starting here.

then solve these next

$$(\underline{\underline{A}} + \underline{\underline{I}}) = \begin{bmatrix} 1 & 1 & 2 \\ -5 & -2 & -7 \\ 1 & 0 & 1 \end{bmatrix} \qquad (\underline{\underline{A}} + \underline{\underline{I}})^2 = \begin{bmatrix} -2 & -1 & -3 \\ -2 & -1 & -3 \\ 2 & 1 & 3 \end{bmatrix}$$

$$(\underline{\underline{A}} + \underline{\underline{I}})^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So, first solve:

$$(\underline{\underline{A}} + \underline{\underline{I}})^3 \underline{u}_3 = \underline{0}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rightarrow$ 3 free variables. Can choose any non-zero vector. choose

$$\underline{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Now plug into

$(\underline{\underline{A}} + \underline{\underline{I}})\,\underline{u}_3 = \underline{u}_2$

$$\begin{bmatrix} 1 & 1 & 2 \\ -5 & -2 & -7 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ 1 \end{bmatrix} \quad \text{so} \quad \underline{u}_2 = \begin{bmatrix} 1 \\ -5 \\ 1 \end{bmatrix}$$

Now plug into

$(\underline{\underline{A}} + \underline{\underline{I}})\,\underline{u}_2 = \underline{u}_1$

$$\begin{bmatrix} 1 & 1 & 2 \\ -5 & -2 & -7 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} 1-5+2 \\ -5+10-7 \\ 1+0+1 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix}$$

so $\underline{u}_1 = \begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix}$ this is the same eigenvector $\underline{v}$ we found earlier